

An infinite family of counterexamples to the Jacobian Conjecture in dimension three: every generic fiber degree $n \geq 3$ occurs

Alexis Gallagher
Repertory Robotics, Inc.

20 July 2026

Abstract

The Jacobian Conjecture [1, 2] asserts that a polynomial map $F: \mathbb{C}^n \rightarrow \mathbb{C}^n$ with $\det JF$ a nonzero constant is invertible. On 19 July 2026, L. Alpöge announced an explicit counterexample in dimension three [3]. Building on it, we give a uniform one-variable construction which, for every integer $n \geq 3$, produces an explicit polynomial map $F_n: \mathbb{C}^3 \rightarrow \mathbb{C}^3$ with $\det JF_n \equiv 1$ and generic fiber degree exactly n . Thus in fixed dimension three, non-injective polynomial maps of constant Jacobian occur in every generic fiber degree $n \geq 3$; the degree-four member is neither coordinate-equivalent to Alpöge's degree-three map nor an iterate of it.

1 The construction

Fix a nonzero constant $c \in \mathbb{C}$ and a *seed* $p \in \mathbb{C}[w]$ with

$$p(0) = 0, \quad p(1) = -c, \quad \int_0^1 p(w) \, dw = 0. \quad (1)$$

Define $q \in \mathbb{C}[w]$ by $q(0) = 0$ and $cq'(w) = wp'(w)$. Put $\kappa = p'(1)/c$, assume $\kappa \neq -2$, and set $a = -(1 + \kappa)/(2 + \kappa)$. Fix a nonzero $b \in \mathbb{C}$. In the variables

$$v = xy, \quad t = x^2z, \quad u = 1 + v, \quad \gamma = 1 + av + bt, \quad w = u\gamma,$$

define $\beta = c + p(w)/\gamma$ and $\alpha = u + q(w)/\gamma^2$, and finally

$$F_p(x, y, z) = \left(\frac{\alpha}{x^2}, \frac{\beta}{x}, x\gamma \right). \quad (2)$$

Theorem 1. F_p is a polynomial map $\mathbb{C}^3 \rightarrow \mathbb{C}^3$ with $\det JF_p = bc$, a nonzero constant.

Proof. Polynomiality. Since $p(0) = 0$ and $w = u\gamma$, we have $p(w)/\gamma = u \cdot (p/w)(w)$, a polynomial in (x, y, z) ; as $q(0) = q'(0) = 0$, likewise $q(w)/\gamma^2 = u^2 \cdot (q/w^2)(w)$ is polynomial. Hence $\alpha, \beta \in \mathbb{C}[x, y, z]$. From (1) one computes $q(1) = \frac{1}{c} \int_0^1 wp'(w) \, dw = \frac{1}{c}(p(1) - \int_0^1 p) = -1$, so $\beta(0, 0) = c + p(1) = 0$ and $\alpha(0, 0) = 1 + q(1) = 0$. Moreover $\alpha_v(0, 0) = 1 + \kappa(1 + a) + 2a = 1 + \kappa + a(\kappa + 2) = 0$ by the choice of a . Thus $\beta \in (v, t)$ and $\alpha \in (v^2, t)$ in $\mathbb{C}[v, t]$. After substituting $v = xy$ and $t = x^2z$, this gives $\beta \in x\mathbb{C}[x, y, z]$ and $\alpha \in x^2\mathbb{C}[x, y, z]$, proving that the first two coordinates of (2) are polynomial.

Constant Jacobian. Write $A = \alpha/x^2$, $B = \beta/x$, $C = x\gamma$ for the coordinates of F_p , and set

$$P := BC = \beta\gamma = c\gamma + p(w), \quad Q := C^2A = \alpha\gamma^2 = w\gamma + q(w).$$

The output change $(A, B, C) \mapsto (P, Q, C)$ has Jacobian $-C^3$. On the other hand factor $(x, y, z) \mapsto (x, v, t) \mapsto (x, w, \gamma)$: the first map has Jacobian x^3 , and $(v, t) \mapsto (w, \gamma)$ has Jacobian $b\gamma$ (from $u = 1 + v$, $\gamma = 1 + av + bt$, $w = u\gamma$). The engine of the whole computation is the one-line identity

$$\frac{\partial(P, Q)}{\partial(w, \gamma)} = \det \begin{pmatrix} p'(w) & c \\ \gamma + q'(w) & w \end{pmatrix} = wp'(w) - c\gamma - cq'(w) = -c\gamma, \quad (3)$$

valid for *every* seed by $cq' = wp'$. Since P, Q depend on (x, y, z) only through (w, γ) while $C = x\gamma$ carries the one $\partial/\partial x$,

$$\frac{\partial(P, Q, C)}{\partial(x, y, z)} = \gamma \cdot \frac{\partial(P, Q)}{\partial(v, t)} \cdot x^3 = \gamma \cdot (-c\gamma)(b\gamma) \cdot x^3 = -bc(x\gamma)^3 = -bcC^3.$$

Comparing with the output change $-C^3 \det JF_p$ gives $\det JF_p = bc$. □

2 The inverse equation and the generic fiber

Let $R(w) = \int_0^w p(s) ds$, so $\deg R = 1 + \deg p$. Eliminating γ from $P = c\gamma + p(w)$, $Q = w\gamma + q(w)$ and using $cq = wp - R$ gives the *inverse equation*

$$R(w) = wP - cQ, \quad P = BC, \quad Q = C^2A. \quad (4)$$

Proposition 1. *If $\deg p = d$, then F_p has generic fiber degree $d + 1$.*

Proof. For a target (A, B, C) with $C \neq 0$, put $f_{P,Q}(w) := R(w) - wP + cQ$, where $P = BC$ and $Q = C^2A$. This polynomial has degree $d + 1$. Its discriminant is not identically zero: indeed, when $P = 0$, a multiple root r would satisfy $p(r) = f'_{0,Q}(r) = 0$ and $Q = -R(r)/c$, excluding only finitely many values of Q . Hence $f_{P,Q}$ has $d + 1$ distinct roots on a nonempty Zariski-open set.

For every such root, $\gamma = (P - p(w))/c = -f'_{P,Q}(w)/c \neq 0$, and it reconstructs a unique input via $u = w/\gamma$, $x = C/\gamma$, $y = (u - 1)/x$, and $z = (\gamma - 1 - a(u - 1))/(bx^2)$. Conversely, every preimage with $C \neq 0$ yields a root of $f_{P,Q}$ by (4). Thus the generic fiber has exactly $d + 1$ points. $\square$

3 Every generic fiber degree at least three, in dimension three

For $n \geq 3$ put $d = n - 1$ and take the single seed

$$p_d(w) = 2w - 3w^2 + w(1 - w)\left(w^{d-2} - \frac{6}{d(d+1)}\right), \quad c = 1. \quad (5)$$

Lemma 1. *For every integer $d \geq 2$ the seed p_d satisfies (1) with $c = 1$, has $\deg p_d = d$, and $p'_d(1) = -5 + \frac{6}{d(d+1)} \in (-5, -4]$, in particular $\neq -2$.*

Proof. Write $p_d = 2w - 3w^2 + g$ with $g(w) = w(1 - w)(w^{d-2} - k)$, $k = \frac{6}{d(d+1)}$. As $g(0) = g(1) = 0$, $p_d(0) = 0$ and $p_d(1) = 2 - 3 = -1$. Since $\int_0^1 (2w - 3w^2) dw = 0$ and

$$\int_0^1 g dw = \int_0^1 (w^{d-1} - w^d) dw - k \int_0^1 (w - w^2) dw = \frac{1}{d(d+1)} - \frac{k}{6} = 0,$$

the integral condition holds — this is exactly what fixes k . Expanding $g = w^{d-1} - w^d - kw + kw^2$: for $d \geq 3$ the top term is $-w^d$, so $\deg p_d = d$ with leading coefficient -1 (for $d = 2$, $k = 1$ and $g \equiv 0$, giving $p_2 = 2w - 3w^2$). Finally $g'(1) = (d - 1) - d - k + 2k = k - 1$, whence $p'_d(1) = -4 + g'(1) = -5 + k$; as $0 < k \leq 1$ this lies in $(-5, -4]$. $\square$

By Lemma 1, Theorem 1 (with $b = c = 1$) and Proposition 1 apply. (For $d = 2$ the seed is $2w - 3w^2$, underlying Alpöge's map.)

Corollary 1. *For every integer $n \geq 3$ the map $F_n := F_{p_{n-1}}$ is an explicit polynomial map $\mathbb{C}^3 \rightarrow \mathbb{C}^3$ with $\det JF_n \equiv 1$ and generic fiber degree n . In particular every geometric degree $n \geq 3$ is realized by a constant-Jacobian polynomial self-map of the fixed space $\mathbb{C}^3$.*

Independently and contemporaneously, C. Aldred [4] announced a different infinite family in $\mathbb{C}^3$, realizing the odd generic fiber degrees $2m - 1$; the construction above realizes every integer $n \geq 3$.

A genuinely distinct member. Take $c = 1$, $p(w) = w - 2w^3$, $a = -\frac{4}{3}$, $b = -1$; after $y \mapsto 3y$ and scaling the first output by 2 one obtains an integer map of component degrees $(12, 11, 4)$, $\det J = -6$, and generic fiber degree 4. Since generic fiber degree is a coordinate-change invariant and 4 is not a power of 3, this map is neither coordinate-equivalent to Alpöge's degree-three example nor an iterate of it.

Code, verification, acknowledgments. Exact symbolic checks (polynomiality, the Jacobian identities, the inverse equation, and exact n -point fibers for $3 \leq n \leq 100$; 5,047 rational preimages) are in the accompanying code (rev. 2fa62681). The seed is due to L. Alpöge, whose announcement credits Fable; this work relied on GPT-5.6-Sol and Fable.

References

- [1] O.-H. Keller, *Ganze Cremona-Transformationen*, Monatsh. Math. Phys. **47** (1939), 299–306.
- [2] H. Bass, E. Connell, D. Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. **7** (1982), 287–330.
- [3] L. Alpöge, announcement, 19 July 2026, https://x.com/__alpoge__/status/2079028340955197566.
- [4] C. Aldred, announcement, 20 July 2026, <https://x.com/CalAldred/status/2079169959972769869>.